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The Complexity of Octopus Graph, Friendship Graph, and Snail Graph

Fransiskus Fran[1], Alexander[2*], Yundari[3], Putri Romanda[4], Ervina Febyolga[5]

[1,2,3,4]Department of Mathematics FMIPA Universitas Tanjungpura

e-mail: ***h1011211025@student.untan.ac.id**

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Abstract

Graphs are basic structures that represent objects with vertices and relationships between objects with edges. Trees are one of the parts studied in graph theory along with finding the complexity of a graph such as octopus graph, friendship graph, and snail graph. The complexity of an octopus graph is strongly dependent on the fan graph that forms the octopus graph, the complexity of a friendship graph is dependent on the number of triangle cycles, and the complexity of a snail graph is dependent on the number of edges and vertices located in the shell-like part of the snail. To calculate the complexity $(\tau(G))$ of a graph, various calculations can be used, such as the extension of Kirchhoff's formula. The extension of Kirchhoff's formula uses the determinant of the adjacency matrix and degree matrix of the graph complement of a graph. Therefore, this research applies the extension of Kirchhoff's formula to obtain the complexity of octopus graph, friendship graph, and snail graph. From the analysis, it is obtained that for $n \geq 2$, the complexity of octopus graph and friendship graph are $\tau(O_n) = \frac{1}{5}\sqrt{5}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right]$ and $\tau(F_n) = 3^n$ and the complexity of snail graph for $n \geq 1$ is $\tau(SI_n) = 2^{n+2} + 3n \cdot 2^{n-1}$.

Keywords: Number of spanning trees, Graph complement, Adjacency matrix, Nonsingular matrix

1. Introduction

Graphs are basic structures in discrete mathematics and computer science that are used to represent various types of networks and relationships. In its application, a graph represents an object with vertices and the relationship between objects with edges (Mujib, 2019). The application of graphs is important because it is useful in various applications, such as search algorithms, electrical circuits, maps, depiction of communication networks, and others. One part of the study of graph theory is about trees and spanning trees of a graph and the formula of finding the number of spanning trees of a graph (Larasati et al., 2023). In various fields of science, the concept of trees is one of the concepts that can support the application of graphs. In 1847, Kirchoff (1824-1887) developed tree theory to be applied in electrical networks (Kirby et al., 2016).

A tree is a connected graph that has no cycles, where a cycle is an alternating sequence of vertices and edges whose starting and ending vertices are the same. If there exists a connected graph G that is not a tree, then a spanning tree of G can be found by removing one or more edges of G without removing the vertices. Then, a spanning tree that contains all the vertices in G can be obtained (Fikadila et al., 2024). At least, the number of spanning trees contained in a connected graph is one. The number of spanning trees of a graph is known as graph complexity. One of the methods that can be used in finding the complexity of a graph without having to describe its spanning trees one by one is Kirchhoff's formula (Ocansey, 2015). In addition, there is another method that uses an extension of Kirchhoff's formula to find the number of spanning trees of a graph (Liu et al., 2021).

Several studies related to graph complexity have been conducted such as determining the number of spanning trees, for example, on a circular graph using the Matrix-Tree Theorem (Fikadila et al., 2024). Another research also discusses the complexity of graphs generated by wheel graphs and their asymptotic limits using the formula which is generalized from the formula of Kelmans and Chelnokov (Daoud, 2017). In addition, finding the number of spanning trees using the extension of Kirchhoff's formula has also been done as done by (Daoud & Mohamed, 2017) discussing the complexity of some families of cycle-related graphs (gear graph, flower graph, sun graph and sphere graph) and (Daoud & Elsonbaty, 2013) which discusses the complexity of some graphs generated by ladder graph. From these studies, it is known that the extension of Kirchhoff's formula has the advantage to express $\tau(G)$ directly as determinant rather than in the form of cofactor in Kirchhoff's formula or eigenvalue in Kelmans and Chelnokov's formula. Another study by (Daoud, 2019) discussing the complexity of some families of graphs generated by a triangle and get the solution by using the recurrence relation. Therefore, in the following, we discuss the complexity of octopus graph, friendship graph, and snail graph using the extension of Kirchhoff's formula and the formula of recurrence relation.

2. Methods

Definition 1. *A graph G is defined as a set pair (V, E), written with the notation $G = \big(V(G), E(G)\big)$, where $V(G)$ is the non-empty set of vertices and $E(G)$ is the set of unordered pairs of vertices called edges* (Rahayuningsih, 2018).

Definition 2. *Two vertices in a graph G are said to be neighbours or adjacencies if they are directly connected by an edge. In other words, vertex u is neighbor to vertex v if there is an edge (uv) in the graph G* (Rahayuningsih, 2018).

Definition 3. *The degree of a vertex in an undirected graph is the number of edges adjacent to that vertex. The degree of a vertex v is expressed by $d(v)$* (Rahayuningsih, 2018).

Definition 4. *An octopus graph is a graph formed from the vertex identification process between a fan graph and a star graph $S_{1,n}$ at the vertex that has the highest degree. Octopus graph is denoted by O_n* (Abdurahman et al., 2024).

Definition 5. *A friendship graph with n vertices denoted by F_n is a graph that has n copies of the graph C_3 that meet at one center vertex with $n \geq 2$* (Putri et al., 2022).

Definition 6. *A snail graph is a graph resulting from the edge identification operation between a triangular book graph and a cycle graph C_4. The snail graph is denoted by SI_n which has $n + 6$ vertices and $2n + 6$ edges* (Komarullah, 2023).

Definition 7. *Suppose $G = (V, E)$ is a simple graph. Complement of graph G denoted $\bar{G} = (V, E^C)$ is a simple graph with edge $(u, v) \in E^c$ if and only if edge $(u, v) \notin E$. That is, the graph G and its graph complement $(\bar{G})$ have the same vertex set, while their edge sets complement each other* (Rahayuningsih, 2018).

Definition 8. *The adjacency matrix of a simple graph with n vertices is a square matrix of size $n \times n$ that expresses the adjacency of vertices in G. Graph G has the adjacency matrix which is expressed by $A = [a_{ij}]; i, j = 1,2, \dots, n$ with $a_{ij} = 1$ if vertex u is neighbor to vertex v in G and $a_{ij} = 0$ if there is no edge connecting vertices u and v* (Fikadila et al., 2024).

Definition 9. *The degree matrix of a graph G with n vertices is a $n \times n$ diagonal matrix and expressed by $D = [a_{ij}]; i, j = 1,2, \dots, n$ with $a_{ij} = d(v_i)$ if $i = j$ and $a_{ij} = 0$ if $i \neq j$* (Fikadila et al., 2024).

Definition 10. *Tree is a connected undirected graph that does not contain circuits or commonly known as a connected graph without cycles* (Priyambodo & Budayasa, 2021).

Definition 11. *The spanning tree of a connected graph G is a subgraph of G that is a tree and connects all vertices of G* (Priyambodo & Budayasa, 2021).

Lemma 1. *Let G be a graph with n vertices. Then,*

$$\tau(G) = \frac{1}{n^2} \det(nI - \bar{D} + \bar{A})$$

where $\bar{A}, \bar{D}$ are the adjacency and degree matrices, respectively, of $\bar{G}$, the complement of G, and I is the $n \times n$ identity matrix (Daoud & Mohamed, 2017).

Lemma 2. *Let A, B, C, and D be matrices of dimensions $n \times n$, $n \times m$, $m \times n$, and $m \times m$, respectively* (Daoud & Mohamed, 2017). *Assume that A and D are nonsingular; then*

$$\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = (-1)^{nm} \det(A - BD^{-1}C) \det D = (-1)^{nm} \det(D - CA^{-1}B) \det A.$$

3. Result and Discussion

The complexity of octopus graph, friendship graph, and snail graph are explained in this following section.

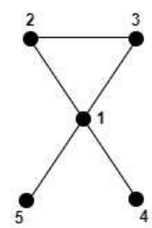
3.1. The Complexity of Octopus Graph (O_n); $n \geq 2$

In this section, we explain the way to find the number of spanning trees of octopus graph by using **Lemma 1**.

The following are the steps to find the number of spanning trees of octopus graph O_2.

Figure 1

Octopus Graph O_2



From Figure 1, we obtain:

$$\bar{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \text{ and } \bar{D} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\tau(G) = \frac{1}{n^2} \det(nI - \bar{D} + \bar{A})$$

$$\tau(O_2) = \frac{1}{5^2}\det\left(\begin{bmatrix}5&0&0&0&0\\0&5&0&0&0\\0&0&5&0&0\\0&0&0&5&0\\0&0&0&0&5\end{bmatrix} - \begin{bmatrix}0&0&0&0&0\\0&2&0&0&0\\0&0&2&0&0\\0&0&0&3&0\\0&0&0&0&3\end{bmatrix} + \begin{bmatrix}0&0&0&0&0\\0&0&0&1&1\\0&0&0&1&1\\0&1&1&0&1\\0&1&1&1&0\end{bmatrix}\right)$$

$$= \frac{1}{25}\det\left(\begin{bmatrix}5&0&0&0&0\\0&3&0&1&1\\0&0&3&1&1\\0&1&1&2&1\\0&1&1&1&2\end{bmatrix}\right) = \frac{5}{25}\det\left(\begin{bmatrix}3&0&1&1\\0&3&1&1\\1&1&2&1\\1&1&1&2\end{bmatrix}\right) = \frac{1}{5}\det\left(\begin{bmatrix}A&B\\B&C\end{bmatrix}\right)$$

$$\tau(O_2) = \frac{1}{5}\det C\,\det(A - BC^{-1}B)$$

$$= \frac{1}{5}\det\left(\begin{bmatrix}2&1\\1&2\end{bmatrix}\right)\det\left(\begin{bmatrix}3&0\\0&3\end{bmatrix} - \begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}\frac{2}{3}&-\frac{1}{3}\\-\frac{1}{3}&\frac{2}{3}\end{bmatrix}\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)$$

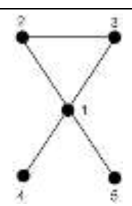
$$= \frac{1}{5}\begin{vmatrix}2&1\\1&2\end{vmatrix}\det\left(\begin{bmatrix}\frac{7}{3}&-\frac{2}{3}\\-\frac{2}{3}&\frac{7}{3}\end{bmatrix}\right) = \frac{1}{5}\cdot 3\cdot\begin{vmatrix}\frac{7}{3}&-\frac{2}{3}\\-\frac{2}{3}&\frac{7}{3}\end{vmatrix} = \frac{3}{5}\cdot 5 = 3$$

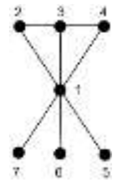
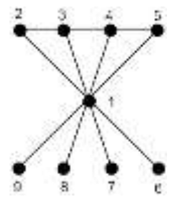
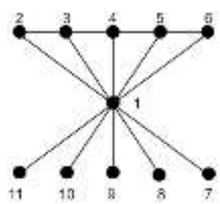
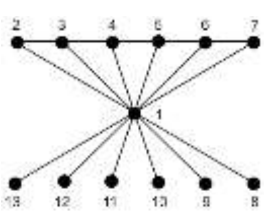
The number of spanning trees of octopus graph O_2 is 3.

By using the same way, the number of spanning trees of octopus graph $O_n; n \geq 2$ can be obtained which can be seen in Table 1.

Table 1

Octopus Graph $O_n; n \geq 2$

n	Octopus Graph O_n	$(nI - \overline{D} + \overline{A})$	$\lvert V(O_n)\rvert$	$\tau(O_n)$
2		$\begin{bmatrix}5&0&0&0&0\\0&3&0&1&1\\0&0&3&1&1\\0&1&1&2&1\\0&1&1&1&2\end{bmatrix}$	5	3

3		$\begin{bmatrix} 7&0&0&0&0&0&0\\ 0&3&0&1&1&1&1\\ 0&0&4&0&1&1&1\\ 0&1&0&3&1&1&1\\ 0&1&1&1&2&1&1\\ 0&1&1&1&1&2&1\\ 0&1&1&1&1&1&2 \end{bmatrix}$	7	8
4		$\begin{bmatrix} 9&0&0&0&0&0&0&0&0\\ 0&3&0&1&1&1&1&1&1\\ 0&0&4&0&1&1&1&1&1\\ 0&1&0&4&0&1&1&1&1\\ 0&1&1&0&3&1&1&1&1\\ 0&1&1&1&1&2&1&1&1\\ 0&1&1&1&1&1&2&1&1\\ 0&1&1&1&1&1&1&2&1\\ 0&1&1&1&1&1&1&1&2 \end{bmatrix}$	9	21
5		$\begin{bmatrix} 11&0&0&0&0&0&0&0&0&0&0\\ 0&3&0&1&1&1&1&1&1&1&1\\ 0&0&4&0&1&1&1&1&1&1&1\\ 0&1&0&4&0&1&1&1&1&1&1\\ 0&1&1&0&4&0&1&1&1&1&1\\ 0&1&1&1&0&3&1&1&1&1&1\\ 0&1&1&1&1&1&2&1&1&1&1\\ 0&1&1&1&1&1&1&2&1&1&1\\ 0&1&1&1&1&1&1&1&2&1&1\\ 0&1&1&1&1&1&1&1&1&2&1\\ 0&1&1&1&1&1&1&1&1&1&2 \end{bmatrix}$	11	55
6		$\begin{bmatrix} 13&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&3&0&1&1&1&1&1&1&1&1&1&1\\ 0&0&4&0&1&1&1&1&1&1&1&1&1\\ 0&1&0&4&0&1&1&1&1&1&1&1&1\\ 0&1&1&0&4&0&1&1&1&1&1&1&1\\ 0&1&1&1&0&4&0&1&1&1&1&1&1\\ 0&1&1&1&1&0&3&1&1&1&1&1&1\\ 0&1&1&1&1&1&1&2&1&1&1&1&1\\ 0&1&1&1&1&1&1&1&2&1&1&1&1\\ 0&1&1&1&1&1&1&1&1&2&1&1&1\\ 0&1&1&1&1&1&1&1&1&1&2&1&1\\ 0&1&1&1&1&1&1&1&1&1&1&2&1\\ 0&1&1&1&1&1&1&1&1&1&1&1&2 \end{bmatrix}$	13	144

Referring to Table 1, for $n \geq 2$, the number of vertices of octopus graph is $2n+1$ and the n^{th} of the $(nI - \overline{D} + \bar{A})$ matrix form is obtained, that is

$$\begin{bmatrix} 2n+1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 3 & 0 & 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ \vdots & 0 & 4 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 4 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & \cdots & \cdots & 1 & 0 & 3 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 2 & 1 & \cdots & \cdots & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & 1 & 2 \end{bmatrix}$$

From the result of the description in Table 1, the number of spanning tree of octopus graph is obtained, as follows: $3,8,21,55,144, \ldots$.

It can be noticed that the results obtained are $3,8,21,55,144, \ldots$ for $n = 2,3,4,5,6, \ldots$. These values form a sequence that can be written in the form:

$$21 = 3(8) - 3,$$
$$55 = 3(21) - 8,$$
$$144 = 3(55) - 21, \ldots$$

From this form, the recurrence relation is obtained:

$$a_n = 3a_{n-1} - a_{n-2}$$

with $a_2 = 3$, $a_3 = 8$ which has a characteristic equation $r^2 - 3r + 1 = 0$ with two different real roots $r_1 = \frac{3+\sqrt{5}}{2}$ and $r_2 = \frac{3-\sqrt{5}}{2}$. So, the recurrence relation has a general solution which is:

$$a_n = \left[\alpha \left(\frac{3+\sqrt{5}}{2}\right)^n + \beta \left(\frac{3-\sqrt{5}}{2}\right)^n\right]$$

By using the initial conditions $a_2 = 3$ and $a_3 = 8$, the values $\alpha = \frac{1}{5}\sqrt{5}$ and $\beta = -\frac{1}{5}\sqrt{5}$ are obtained so that:

$$a_n = \frac{1}{5}\sqrt{5}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right]$$

Based on the above formulation, it is obtained that for $n \geq 2$, the complexity of octopus graph is expressed by:

$$\tau(O_n) = \frac{1}{5}\sqrt{5}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right]$$

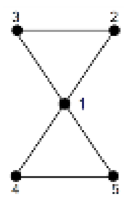
3.2. The Complexity of Friendship Graph $(F_n); n \geq 2$

In this section, we explain the way to find the number of spanning trees of friendship graph by using **Lemma 1**.

The following are the steps to find the number of spanning trees of friendship graph F_2.

Figure 2

Friendship Graph (F_2)



From Figure 2, we obtain:

$$\bar{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \text{ and } \overline{D} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\tau(G) = \frac{1}{n^2}\det(nI - \overline{D} + \bar{A})$$

$$\tau(F_2) = \frac{1}{5^2}\det\left(\begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}\right)$$

$$= \frac{1}{25}\det\left(\begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 & 3 \end{bmatrix}\right) = \frac{5}{25}\det\left(\begin{bmatrix} 3 & 0 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 3 \end{bmatrix}\right) = \frac{1}{5}\det\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right)$$

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$$= \frac{1}{5}\det A \det(A - BA^{-1}B) = \frac{1}{5}\det\left(\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}\right)\det\left(\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix} - \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{3} & 0\\0 & \frac{1}{3}\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\right)$$

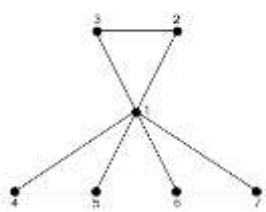
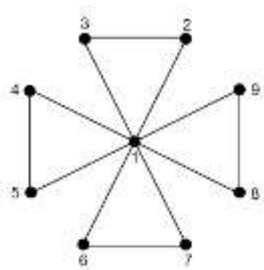
$$\tau(F_2) = \frac{1}{5}\begin{vmatrix}3 & 0\\0 & 3\end{vmatrix}\det\left(\begin{bmatrix}\frac{7}{3} & -\frac{2}{3}\\-\frac{2}{3} & \frac{7}{3}\end{bmatrix}\right) = \frac{1}{5}\cdot 3^2 \cdot \begin{vmatrix}\frac{7}{3} & -\frac{2}{3}\\-\frac{2}{3} & \frac{7}{3}\end{vmatrix} = \frac{3^2}{5}\cdot 5 = 3^2$$

The number of spanning trees of friendship graph F_2 is 3^2.

By using the same way, the number of spanning trees of friendship graph $F_n; n \geq 2$ can be obtained which can be seen in Table 2.

Table 2

Friendship Graph $F_n; n \geq 2$

n	Friendship Graph F_n	$(nI - \overline{D} + \overline{A})$	$\lvert V(F_n)\rvert$	$\tau(F_n)$
2		$\begin{bmatrix}5&0&0&0&0\\0&3&0&1&1\\0&0&3&1&1\\0&1&1&3&0\\0&1&1&0&3\end{bmatrix}$	5	3^2
3		$\begin{bmatrix}7&0&0&0&0&0&0\\0&3&0&1&1&1&1\\0&0&3&1&1&1&1\\0&1&1&3&0&1&1\\0&1&1&0&3&1&1\\0&1&1&1&1&3&0\\0&1&1&1&1&0&3\end{bmatrix}$	7	3^3
4		$\begin{bmatrix}9&0&0&0&0&0&0&0&0\\0&3&0&1&1&1&1&1&1\\0&0&3&1&1&1&1&1&1\\0&1&1&3&0&1&1&1&1\\0&1&1&0&3&1&1&1&1\\0&1&1&1&1&3&0&1&1\\0&1&1&1&1&0&3&1&1\\0&1&1&1&1&1&1&3&0\\0&1&1&1&1&1&1&0&3\end{bmatrix}$	9	3^4

Referring to Table 2, for $n \geq 2$, the number of vertices of friendship graph is $2n + 1$ and the n^{th} of the $(nI - \overline{D} + \bar{A})$ matrix form is obtained, for n odd is:

$$\begin{bmatrix} 2n+1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 3 & 0 & 1 & \cdots & 1 & 1 & \cdots & \cdots & \cdots & 1 \\ \vdots & 0 & \ddots & 1 & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 1 & 1 & \ddots & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & \ddots & 1 & 1 & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & \cdots & 1 & 1 & 3 & 0 & 1 & \cdots & \cdots & 1 \\ 0 & 1 & \cdots & \cdots & 1 & 0 & 3 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 1 & 1 & \ddots & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & 1 & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & 1 & \ddots & 0 \\ 0 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & 1 & 0 & 3 \end{bmatrix}$$

And the n^{th} of the $(nI - \overline{D} + \bar{A})$ matrix form is obtained, for n even is:

$$\begin{bmatrix} 2n+1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 3 & 0 & 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & 1 & 1 \\ \vdots & 0 & \ddots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 1 & 1 & \ddots & 0 & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & \ddots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & \cdots & \cdots & 1 & 0 & 3 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 3 & 0 & 1 & \cdots & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 1 & \ddots & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & 0 & \ddots & 1 & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \ddots & 0 \\ 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 & 0 & 3 \end{bmatrix}$$

From the result of the description in Table 2, the number of spanning trees of friendship graph is obtained, as follows: $3^2, 3^3, 3^4, \ldots$

It can be noticed that the results obtained are $3^2, 3^3, 3^4, \ldots$ for $n = 2,3,4, \ldots$. These values can be written in the form $3^n; n \geq 2$.

Based on the above formulation, it is obtained that for $n \geq 2$, the complexity of friendship graph is expressed by:

$$\tau(F_n) = 3^n$$

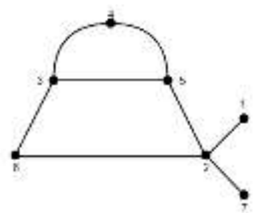
3.3. The Complexity of Snail Graph $(SI_n); n \geq 1$

In this section, we explain the way to find the number of spanning trees of snail graph by using **Lemma 1**.

The following are the steps to find the number of spanning trees of snail graph SI_1.

Figure 3

Snail Graph (SI_1)



From Figure 3, we obtain:

$$\bar{A} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \text{ and } \bar{D} = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

$$\tau(G) = \frac{1}{n^2} \det(nI - \bar{D} + \bar{A})$$

$$\tau(SI_1) = \frac{1}{7^2}\det\left(\begin{bmatrix}7&0&0&0&0&0&0\\0&7&0&0&0&0&0\\0&0&7&0&0&0&0\\0&0&0&7&0&0&0\\0&0&0&0&7&0&0\\0&0&0&0&0&7&0\\0&0&0&0&0&0&7\end{bmatrix}-\begin{bmatrix}5&0&0&0&0&0&0\\0&2&0&0&0&0&0\\0&0&3&0&0&0&0\\0&0&0&4&0&0&0\\0&0&0&0&3&0&0\\0&0&0&0&0&4&0\\0&0&0&0&0&0&5\end{bmatrix}\right.$$

$$\left.+\begin{bmatrix}0&0&1&1&1&1&1\\0&0&1&1&0&0&0\\1&1&0&0&0&0&1\\1&1&0&0&0&1&1\\1&0&0&0&0&1&1\\1&0&0&1&1&0&1\\1&0&1&1&1&1&0\end{bmatrix}\right)$$

$$\tau(SI_1) = \frac{1}{49}\det\left(\begin{bmatrix}2&0&1&1&1&1&1\\0&5&1&1&0&0&0\\1&1&4&0&0&0&1\\1&1&0&3&0&1&1\\1&0&0&0&4&1&1\\1&0&0&1&1&3&1\\1&0&1&1&1&1&2\end{bmatrix}\right) = \frac{1}{49}\det\left(\begin{matrix}\begin{bmatrix}2&0&1&1\\0&5&1&1\\1&1&4&0\\1&1&0&3\end{bmatrix} & \begin{bmatrix}1&1&1\\0&0&0\\0&0&1\\0&1&1\end{bmatrix}\\ \begin{bmatrix}1&0&0&0\\1&0&0&1\\1&0&1&1\end{bmatrix} & \begin{bmatrix}4&1&1\\1&3&1\\1&1&2\end{bmatrix}\end{matrix}\right)$$

$$= \frac{1}{49}\det\left(\begin{bmatrix}A&B\\C&D\end{bmatrix}\right) = \frac{1}{49}\det D\,\det(A - BD^{-1}C)$$

$$= \frac{1}{49}\det\left(\begin{bmatrix}4&1&1\\1&3&1\\1&1&2\end{bmatrix}\right)\det\left(\begin{bmatrix}2&0&1&1\\0&5&1&1\\1&1&4&0\\1&1&0&3\end{bmatrix}\right.$$

$$\left.-\begin{bmatrix}1&1&1\\0&0&0\\0&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}\frac{5}{17}&-\frac{1}{17}&-\frac{2}{17}\\-\frac{1}{17}&\frac{7}{17}&-\frac{3}{17}\\-\frac{2}{17}&-\frac{3}{17}&\frac{11}{17}\end{bmatrix}\begin{bmatrix}1&0&0&0\\1&0&0&1\\1&0&1&1\end{bmatrix}\right)$$

$$= \frac{1}{49}\begin{vmatrix} 4 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{vmatrix} \det\left(\begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 5 & 1 & 1 \\ 1 & 1 & 4 & 0 \\ 1 & 1 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \frac{11}{17} & 0 & \frac{6}{17} & \frac{9}{17} \\ 0 & 0 & 0 & 0 \\ \frac{6}{17} & 0 & \frac{11}{17} & \frac{8}{17} \\ \frac{9}{17} & 0 & \frac{8}{17} & \frac{12}{17} \end{bmatrix}\right)$$

$$= \frac{1}{49} \cdot 17 \cdot \det\left(\begin{bmatrix} \frac{23}{17} & 0 & \frac{11}{17} & \frac{8}{17} \\ 0 & 5 & 1 & 1 \\ \frac{11}{17} & 1 & \frac{57}{17} & -\frac{8}{17} \\ \frac{8}{17} & 1 & -\frac{8}{17} & \frac{39}{17} \end{bmatrix}\right) = \frac{17}{49} \cdot \begin{vmatrix} \frac{23}{17} & 0 & \frac{11}{17} & \frac{8}{17} \\ 0 & 5 & 1 & 1 \\ \frac{11}{17} & 1 & \frac{57}{17} & -\frac{8}{17} \\ \frac{8}{17} & 1 & -\frac{8}{17} & \frac{39}{17} \end{vmatrix} = \frac{17}{49} \cdot \frac{539}{17}$$

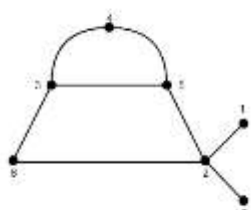
$$= \frac{539}{49} = 11$$

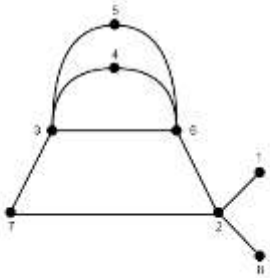
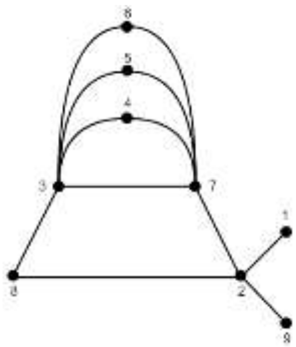
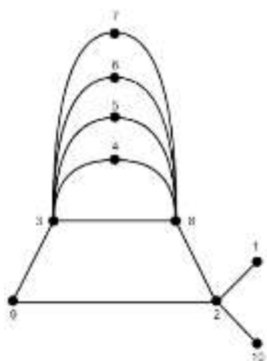
The number of spanning trees of snail graph SI_1 is 11.

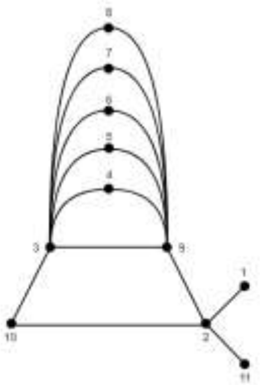
By using the same way, the number of spanning trees of snail graph $SI_n; n \geq 1$ can be obtained which can be seen in Table 3.

Table 3

Snail Graph $(SI_n); n \geq 1$

n	Snail Graph SI_n	$(nI - \overline{D} + \overline{A})$	$\lvert V(SI_n) \rvert$	$\tau(SI_n)$
1		$\begin{bmatrix} 2 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 5 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 4 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 3 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 4 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}$	7	11

2		$\begin{bmatrix} 2 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 5 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 5 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 3 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 3 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 5 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}$	8	28
3		$\begin{bmatrix} 2 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 5 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 6 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 3 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 3 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 3 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 6 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}$	9	68
4		$\begin{bmatrix} 2 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 5 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 3 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 3 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 3 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 3 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}$	10	160

5		$\begin{bmatrix} 2 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 5 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 3 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 3 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 3 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 3 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 3 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}$	11	368

Referring to Table 3, for $n \geq 1$, the number of vertices of snail graph is $n + 6$ and the n^{th} of the $(nI - \overline{D} + \bar{A})$ matrix form is obtained, that is

$$\begin{bmatrix} 2 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 1 & 1 \\ 0 & 5 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 0 & 0 \\ 1 & 1 & n+3 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 1 \\ \vdots & \vdots & 0 & 3 & 1 & \cdots & \cdots & 1 & \vdots & 1 & \vdots \\ \vdots & \vdots & \vdots & 1 & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 1 & \vdots & \vdots & \vdots \\ \vdots & 1 & \vdots & 1 & \cdots & \cdots & 1 & 3 & 0 & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & n+3 & 1 & 1 \\ 1 & 0 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 3 & 1 \\ 1 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 1 & 2 \end{bmatrix}$$

From the result of the description in Table 3, the number of spanning trees of snail graph is obtained, as follows: $11, 28, 68, 160, 368, \ldots$

It can be noticed that the results obtained are $11, 28, 68, 160, 368, \ldots$ for $n = 1, 2, 3, 4, 5, \ldots$. These values form a sequence that can be written in the form:

$$68 = 4(28) - 4(11),$$
$$160 = 4(68) - 4(28),$$
$$368 = 4(160) - 4(68), \ldots$$

From this form, the recurrence relation is obtained:

$$a_n = 4a_{n-1} - 4a_{n-2}$$

with $a_1 = 11$, $a_2 = 28$ which has a characteristic equation $r^2 - 4r + 4 = 0$ with two real and equal roots $r_1 = r_2 = 2$. So, the recurrence relation has a general solution which is:

$$a_n = \alpha \cdot 2^n + \beta \cdot n2^n$$

By using the initial conditions $a_1 = 11$ and $a_2 = 28$, the values $\alpha = 4$ and $\beta = \frac{3}{2}$ are obtained so that:

$$a_n = 4 \cdot 2^n + \frac{3}{2} \cdot n2^n = 2^{n+2} + 3n \cdot 2^{n-1}$$

Based on the above formulation, it is obtained that for $n \geq 1$, the complexity of snail graph is expressed by:

$$\tau(SI_n) = 2^{n+2} + 3n \cdot 2^{n-1}$$

4. Conclusion

From the explanation above, the result of finding the complexity of octopus graph, friendship graph, and snail graph shows that the number of spanning trees of octopus graph, friendship graph, and snail graph are:

1. $\tau(O_n) = \frac{1}{5}\sqrt{5}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right]; n \geq 2$
2. $\tau(F_n) = 3^n; n \geq 2$
3. $\tau(SI_n) = 2^{n+2} + 3n \cdot 2^{n-1}; n \geq 1$

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