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An Exact Symplectic Structure of Low Dimensional 2-Step Solvable Lie Algebras

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Abstract

We introduce another type of a Lie algebra which is equipped by an exact symplectic structure. The dimension of the Lie algebra equipped by this condition is always even. The research aims to identify and to contruct 2-step solvable exact symplectic Lie algebras of low dimension with explicit formulas for their one-forms and symplectic forms. For case of four-dimensional, we found that only one class among three classes is 2-step solvable exact symplectic Lie algebra. Furthermore, we also give more examples for case six and eight dimensional of Lie algebras with exact symplectic forms which is included 2-step solvable exact sympletic Lie algebras. Moreover, we have known well that a Lie algebra with 2-step solvable equipped by an exact symplectic form is nothing but it is called a 2-step Frobenius solvable Lie algebra.

Keywords: A Lie algebra with 2-step solvable, Exact symplectic Lie algebra, One form, Symplectic form

1. Introduction

Let $\mathfrak{g}$ be a vector space, considered it over the field of real numbers $\mathbb{R}$. We call $\mathfrak{g}$ as a Lie algebra if $\mathfrak{g}$ has a Lie bracket with a Jacobi condition. The notion of a symplectic Lie algebra with exact condition is nothing but it is a Frobenius Lie algebra or FLA. Let α be a Frobenius functional in $\mathfrak{g}$ with the property that a bilinear form β on $\mathfrak{g}$ defined by

$$\alpha([k_1, k_2]) = \beta(k_1, k_2), \forall k_1, k_2 \in \mathfrak{g} \quad (1)$$

is symplectic. The FLA $\mathfrak{g}$ is a pair of $\mathfrak{g}$ and α with α is defined in equation (1) (see (Ooms, 1974, 1976, 1980, 2009)).

In Lie algebra theory, we have known well that every nilpotent Lie algebra is a solvable Lie algebra but the converse is not true. An interesting property shows that an FLA is never nilpotent but a solvability of Frobenius Lie algebra attracts many researchers to study more. One of the properties of Frobeius Lie algebras is the notion of 2-step solvable exact symplectic Lie algebra (Diatta et al., 2020). It is interesting to observe that every exact symplectic Lie algebra wich is 2-solvable can be obtained from a set of matrices $M \subseteq M(n, \mathbb{R})$ which is commutative each other and linearly independent with certain conditions. In this case, $M(n, \mathbb{R})$ is a space of real matrices with dimension $n \times n$. We apply this result for low dimensional Lie algebras.

We start to give some examples of $2, 4, 6,$ and 8 dimensional of 2-step solvable exact symplectic Lie algebras (Csikós & Verhóczki, 2007; Ooms, 1980). As mentioned above that their constructions will involve the commutative-linearly independent matrices $M \subseteq M(n, \mathbb{R})$ where $n = 2, 4, 6, 8$. The method is motivated by Diatta works (Diatta et al., 2020). The classification of low dimensional symplectic Lie algebras with exact condition can be studied more for example in Csicoz and Ooms' work(Csikós & Verhóczki, 2007; Ooms, 1980). Some examples of exact symplectic Lie algebras can be found in (Alvarez et al., 2018; Csikós & Verhóczki, 2007; Diatta et al., 2020; Diatta & Manga, 2014; Elashvili, 1983; Gerstenhaber & Giaquinto, 2009; Kurniadi et al., 2021; Ooms, 1980, 2009; Pham, 2016). Different from previous results, in this research we shall apply adjoint representations of a Lie algebra to compute a part of a semi-direct sum of 2-step solvable exact symplectic Lie algebra.

2. Methods

A Frobenius Lie algebra has always symplectic structure. This is why a Frobenius Lie algebra has even dimension. We recall some notion of exact symplectic Lie algebras, 2-step exact symplectic Lie algebra, and their properties.

Definition 1. *A symplectic Lie algebra with exact condition or an FLA is a pair* $(\mathfrak{g}, \alpha)$ *with* $\mathfrak{g}$ *is a Lie algebra and* α *is a Frobenius functional defined in equation* (1).

Example 2. A familiar example of an exact symplectic Lie algebra is the affine Lie algebra of dimension $n(n+1)$. This Lie algebra is denoted by $\mathfrak{aff}(n, \mathbb{R})$ and realized in the following form:

$$\mathfrak{aff}(n, \mathbb{R}) = \left\{ \begin{bmatrix} A & x \\ 0 & 0 \end{bmatrix} ; A \in M_n(\mathbb{R}), x \in \mathbb{R}^n \right\} \subseteq \mathfrak{gl}_{n+1}(\mathbb{R}) \qquad (2)$$

where $\mathfrak{gl}_{n+1}(\mathbb{R})$ is a Lie algebra of $M(n+1, \mathbb{R})$. Let $B = (x_{pq}), 1 \leq p \leq n,$ and $1 \leq q \leq n+1$ be a basis for $\mathfrak{aff}(n, \mathbb{R})$ then a Frobenius functional for $\mathfrak{aff}(n, \mathbb{R})$ can be written in the form

$$\alpha = x_{12}^* + x_{23}^* + \cdots + x_{n,n+1}^* \qquad (3)$$

where $x_{pq}^*, 1 \leq p \leq n,$ and $1 \leq q \leq n+1$ are elements of dual space $\mathfrak{aff}^*(n, \mathbb{R})$.

Moreover, let $\mathfrak{g}$ be an exact symplectic Lie algebra of dimension $2n$. Let $\{k_1, k_2, \dots, k_n, l_1, l_2, \dots, l_n\}$ be basis for $\mathfrak{g}$. Another researcher defines a non-degeneracy of an alternating bilinear form as follows:

$$(\mathrm{d}\alpha)^n = \beta^n = \left(\sum_{i=1}^{n} k_i^* \wedge l_i^* \right)^n \qquad (4)$$

$$= (-1)^{\frac{n(n-1)}{2}} n!\,(k_1^* \wedge k_2^* \dots \wedge k_n^*) \wedge (l_1^* \wedge l_2^* \wedge \dots \wedge l_n^*).$$

Definition 3. *An symplectic Lie algebra $\mathfrak{L}$ with exact condition is said to be a 2-step solvable if its ideal derivation of $\mathfrak{L}$ is commutative. In other words, we have $\big[[\mathfrak{L},\mathfrak{L}],[\mathfrak{L},\mathfrak{L}]\big] = 0$ or $\big[[a,b],[a',b']\big] = 0$ is true for all $a,b,a',b' \in \mathfrak{L}$.*

We recall the result of a 2-step solvable exact symplectic Lie algebra as follows (Diatta et al., 2020):

Theorem 4. *Let $M(n,\mathbb{R})$ be a space of real matrices with dimension $n \times n$ whose Lie algebra is $\mathfrak{gl}_n(\mathbb{R})$. The Lie brackets on $\mathfrak{gl}_n(\mathbb{R})$ are given by $[A,B] = AB - BA$ for each $A,B \in \mathfrak{g}$. Let $M = \{M_1, M_2, \dots, M_n\} \subseteq \mathfrak{gl}_n(\mathbb{R})$ be a linearly independent set and commutative real matrices of size $n \times n$. Let the vector space $V = \langle M \rangle$ acts on $(\mathbb{R}^n)^*$ via ψ^* of the action ψ which is given by $\psi(A)x = Ax, A \in M, x \in \mathbb{R}^n$. Let $\mathfrak{g} = \mathbb{R}^n \oplus V$ be a constructed Lie algebra whose Lie brackets are $[A,B] = [p,q] = 0$ and $[A,q] = \psi(A)q = Aq, A,B \in V, p,q \in \mathbb{R}^n$. Then $\mathfrak{g}$ is a 2-step symplectic Lie algebra with exact condition. We also can write $\mathfrak{g}$ as $\mathfrak{g} = \mathbb{R}^n \rtimes V$ which is a semi-direct sum of $\mathbb{R}^n$ and V.*

Moreover, the constructions of elements of V can be computed by using the adjoint representations of restricted by $\mathbb{R}^n$. Meanwhile, the space $\mathbb{R}^n$ can be constructed as its derived ideal that is $\mathbb{R}^n = [\mathfrak{g},\mathfrak{g}]$. We follow this construction to compute some examples of 2-step exact symplectic Lie algebra. Each 2-step symplectic Lie algebra with exact condition is isomorphic to 2-step exact symplectic Lie algebra which is obtained in this way.

3. Result and Discussion

The first result starts from dimension 2. There is one class of 2-dimensional exact symplectic Lie algebra. The result state as follows:

Proposition 5. *The Lie algebra $\mathfrak{aff}(1,\mathbb{R}) = \langle x_1, x_2 \rangle$ whose Lie bracket is $[x_1,x_2] = x_2$ is 2-step solvable exact symplectic Lie algebra. Its Frobenius functional α is x_2^* and symplectic form β is $x_1^* \wedge x_2^*$. Moreover, $\mathfrak{aff}(1,\mathbb{R}) = \mathbb{R} \rtimes V$ where $V = \langle M \rangle = \mathbb{R}$ and $M = \{a_1 = -1\}$.*

Proof. Since $[x_1,x_2] = x_2$, then $[\mathfrak{g},\mathfrak{g}] = \langle x_2 \rangle$. Indeed, we have $\big[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]\big] = 0$. Thus, $\mathfrak{g}$ is 2-step exact symplectic Lie algebra. Furthermore, the alternating bilinear form defined by $x_2^*([k,l]) = (x_1^* \wedge x_2^*)(k,l)$ is non-degenerate for all $k,l \in \mathfrak{aff}(1,\mathbb{R})$. Namely, we have $\alpha = x_2^*$ and $\beta = x_1^* \wedge x_2^*$. Setting $\mathbb{R} = [\mathfrak{g},\mathfrak{g}] = \langle x_2 \rangle$ and $V = \langle M \rangle = \mathbb{R}$ where $M = \{a_1\}$. In this case $a_1 = \mathrm{ad}(x_1)_{\mathbb{R}} = -1$. Therefore, $\mathfrak{aff}(1,\mathbb{R}) = \mathbb{R} \rtimes V$ where $V = \langle M \rangle = \mathbb{R}$ and $M = \{a_1 = -1\}$. In other words, $\mathfrak{aff}(1,\mathbb{R}) = \mathbb{R} \rtimes \mathbb{R}$.

Let us consider for case 4-dimensional exact symplectic Lie algebra. We first follow the result of Ooms work (Ooms, 1974). There are three types of exact symplectic Lie algebras with the following Lie brackets:

1. $\mathfrak{g}_{4,1} = \mathfrak{aff}(1,\mathbb{R}) \oplus \mathfrak{aff}(1,\mathbb{R}) = \langle p_1, p_2 \rangle \oplus \langle p_1, p_2 \rangle$. The following non-zero Lie brackets of $\mathfrak{g}_{4,1}$ are given by:

$$[(p_1,p_1),(p_2,p_1)] = (p_2,0)$$

$$[(p_1,p_1),(p_2,p_2)] = (p_2,p_2)$$

$$[(p_1,p_2),(p_2,p_1)] = (p_2,-p_2). \quad (4)$$

2. $\mathfrak{g}_{4,2}^{k} = \langle p,q,r,s \rangle$ with Lie brackets are

$[p,q] = kq,$

$[p,r] = (1-k)r,$

$[p,s] = s$, and

$[q,r] = s, \quad (5)$

with $k \in \mathbb{R}$. The Lie algebra $\mathfrak{g}_{4,2}^{a}$ is isomorphic to $\mathfrak{g}_{4,2}^{b}$ if and only if $a = b$ or $a + b = 1$.

3. $\mathfrak{g}_{4,3} = \langle p,q,r,s \rangle$ with Lie brackets are

$$[p,q] = \frac{1}{2}q + r,$$

$$[p,r] = \frac{1}{2}r,$$

$[p,s] = s$, and

$[q,r] = s. \quad (6)$

Proposition 6. *The Lie algebra* $\mathfrak{g}_{4,1} = \mathfrak{aff}(1,\mathbb{R}) \oplus \mathfrak{aff}(1,\mathbb{R})$ *is a* 2*-step symplectic Lie algebra with exact condition. Meanwhile, The Lie algebras* $\mathfrak{g}_{4,2}^{a}$ *and* $\mathfrak{g}_{4,3}$ *are not* 2*-step exact symplectic Lie algebras.*

Proof. We see that the derived ideal of $\mathfrak{g}_{4,1}$ is spanned by $\langle (p_2,p_2) \rangle$, namely $[\mathfrak{g}_{4,1},\mathfrak{g}_{4,1}] = \langle (p_2,p_2) \rangle$. It is easy to observe that $\big[[\mathfrak{g}_{4,1},\mathfrak{g}_{4,1}],[\mathfrak{g}_{4,1},\mathfrak{g}_{4,1}]\big] = O$. Thus, $\mathfrak{g}_{4,1}$ is 2-step exact symplectic Lie algebra. In the other hand, the symplectic Lie algebra $\mathfrak{g}_{4,2}^{a}$ with exact condition is not 2-step exact symplectic Lie algebra since its derived ideal $[\mathfrak{g}_{4,2}^{a},\mathfrak{g}_{4,2}^{a}]$ is not commutative. We have $[q, r] = s$. In addition, the Lie algebra $\mathfrak{g}_{4,3}$ is not 2-step exact symplectic Lie algebra since $[[\mathfrak{g}_{4,3}, \mathfrak{g}_{4,3}], [\mathfrak{g}_{4,3}, \mathfrak{g}_{4,3}]] = \langle s \rangle$.

Let us consider for case 4-dimensional exact symplectic Lie algebra. The second case we follow the result of Csicoz's work (Csikós & Verhóczki, 2007). There are three types of symplectic Lie algebras with exact condition equipped by the following non-zero Lie brackets:

1. $\mathfrak{h}_{4,1} = \langle w_1, w_2, w_3, w_4 \rangle$. The non-zero Lie brackets of $\mathfrak{g}_{4,1}$ are given by:

$$[w_1, w_4] = -w_1 = [w_2, w_3]$$
$$[w_2, w_4] = -\frac{w_2}{2}$$
$$[w_3, w_4] = -\frac{w_3}{2} \tag{7}$$

2. $\mathfrak{h}_{4,2}^k = \langle w_1, w_2, w_3, w_4 \rangle$ with Lie brackets are

$$[w_1, w_4] = -w_1 = [w_2, w_3]$$
$$[w_2, w_4] = -w_3$$
$$[w_3, w_4] = -w_3 + kw_2 \tag{8}$$

with $k \in \mathbb{R}$.

3. $\mathfrak{h}_{4,3}^l = \langle w_1, w_2, w_3, w_4 \rangle$ with Lie brackets are

$$[w_1, w_3] = [w_2, w_4] = -w_1$$
$$[w_1, w_4] = lw_2$$
$$[w_2, w_3] = -w_2 \tag{9}$$

with $l \in \mathbb{R}|\{0\}$.

The result stated in the following proposition:

Proposition 7. *The Lie algebra $\mathfrak{h}_{4,3}^l$ is a 2-step symplectic Lie algebra with exact condition. Moreover, it can be realized as a semidirect sum $\mathfrak{h}_{4,3}^l = \mathbb{R}^2 \rtimes V = \langle w_1, w_2 \rangle \rtimes \langle A_1, A_2 \rangle$ where $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 1 \\ -l & 0 \end{bmatrix}$. Furthermore, a Frobenius functional of $\mathfrak{h}_{4,3}^l$ is of the form $\pm a_i^*, i = 1,2$. Meanwhile, the Lie algebras $\mathfrak{h}_{4,1}$ and $\mathfrak{h}_{4,2}^k$ are not 2-step exact symplectic Lie algebras.*

Proof. We see that its derived ideal of $\mathfrak{h}_{4,3}^l$ is $[\mathfrak{h}_{4,3}^l, \mathfrak{h}_{4,3}^l] = \langle w_1, w_2 \rangle$. Direct computations we also have that $[\mathfrak{h}_{4,3}^l, \mathfrak{h}_{4,3}^l]$ is commutative. Then $\mathfrak{h}_{4,3}^l$ is 2-step exact symplectic Lie algebra. Furthermore, we can choose $\mathbb{R}^2 = [\mathfrak{h}_{4,3}^l, \mathfrak{h}_{4,3}^l] = \langle w_1, w_2 \rangle$. We claim as follows. Let V be spanned by A_1 and A_2 where $M = \{A_1, A_2\} \subseteq M(2,\mathbb{R})$ is linearly independent and commutating 2×2 matrices such that $\mathfrak{h}_{4,3}^{l} = \mathbb{R}^2 \rtimes V = \langle w_1, w_2 \rangle \rtimes \langle A_1, A_2 \rangle$ where $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 1 \\ -l & 0 \end{bmatrix}$. To prove this, let $\mathrm{ad}|_{\mathbb{R}^2}: \langle w_3, w_4 \rangle \to \mathfrak{gl}(\mathfrak{h}_{4,3}^{l})$ be an adjoint representation of $\langle w_3, w_4 \rangle$ on the space $\mathfrak{h}_{4,3}^{l}$. Then by direct computations and by Theorem 4 we have:

$$\mathrm{A}_1 = \mathrm{ad}(w_3)|_{\mathbb{R}^2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\mathrm{A}_2 = \mathrm{ad}(w_4)|_{\mathbb{R}^2} = \begin{bmatrix} 0 & 1 \\ -l & 0 \end{bmatrix}, 0 \neq l \in \mathbb{R}.$$

It is easy to observe that $A_1A_2 = A_2A_1$. Moreover, the linear equation system $\alpha_1 A_1 + \alpha_2 A_2 = O$ is linearly independent since the equation is just satisfied by $\alpha_1 = 0$ and $\alpha_2 = 0$. Therefore, $M = \{A_1, A_2\}$ is linearly independent set and commutative matrices. Following Theorem 4, the Lie algebra constructed in this way is 2-step solvable symplectic Lie algebra with exact condition.

On the other hand, both derived Lie algebra of $\mathfrak{h}_{4,1}$ and $\mathfrak{h}_{4,2}^{k}$ are not commutative. These imply that $\mathfrak{h}_{4,1}$ and $\mathfrak{h}_{4,2}^{k}$ are not 2-step solvable exact symplectic Lie algebra.

■

We are moving the case of 6-dimensional exact symplectic Lie algebra as classified by Csicoz. There classes of 6-dimensional symplectic Lie algebras with exact condition are 16. We investigate for a familiar example of non solvable exact symplectic Lie algebra which is called the 6-dimensional affine Lie algebra. We denote it by $\mathfrak{aff}(2,\mathbb{R}) = \langle h_1, x_1, y_1, a_1, b_1, c_1 \rangle$. Their Lie Lie brackets are given by the following forms:

$$[h_1, x_1] = 2x_1, [h_1, y_1] = -2y_1, [x_1, y_1] = h_1,$$

$$[a_1, b_1] = b_1, [a_1, c_1] = c_1,$$

$$[h_1, b_1] = b_1, [h_1, c_1] = -c_1,$$

$$[x_1, c_1] = b_1, [y_1, b_1] = c_1. \quad (10)$$

We see that its derived ideal is $[\mathfrak{aff}(2,\mathbb{R}), \mathfrak{aff}(2,\mathbb{R})] = \langle x_1, y_1, h_1, b_1, c_1 \rangle$. The derived ideal $[\mathfrak{aff}(2,\mathbb{R}), \mathfrak{aff}(2,\mathbb{R})]$ is not commutative since we find $[h_1, x_1] = 2x_1$. Thus, $\mathfrak{aff}(2,\mathbb{R})$ is not 2-step exact symplectic Lie algebra. Using the non-zero brackets for 6-dimensional exact symplectic Lie algebras in Csicoz work we have that all Lie algebras are not 2-step solvable Lie algebras.

The case of the 8-dimensional 2-step solvable symplectic Lie algebra with exact condition is obtained from the work (Diatta et al., 2020) in **example 4.1.** We recall as follows:

Proposition 8 (Diatta et al., 2020). *Let $\mathfrak{h}$ be a Lie algebra over $\mathbb{R}$ with basis $B = \{h_1, h_2, \ldots, h_8\}$ and non-zero Lie brackets*

$$[h_1, h_5] = h_5, [h_1, h_6] = h_6 = [h_2, h_5]$$

$[h_2, h_6] = -h_5, [h_3, h_7] = h_7$
$[h_3, h_8] = h_8 = [h_4, h_7]$
$[h_4, h_8] = -h_7.$ (11)

Then $\mathfrak{h}$ is 2-step solvable symplectic Lie algebra with exact condition written by

$$\mathfrak{h} = \langle h_5, h_6, h_7, h_8 \rangle \rtimes \langle N_1, N_2, N_3, N_4 \rangle = \mathbb{R}^4 \rtimes N$$

where

$$N_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, N_2 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
$$N_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, N_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

In addition, a Frobenius functional of $\mathfrak{h}$ is of a form $\alpha = h_5^* + h_7^*$ and its symplectic form is $\beta = \partial\alpha = h_2^* \wedge h_6^* + h_5^* \wedge h_1^* + h_7^* \wedge h_3^* + h_8^* \wedge h_4^*$.

Proof. The ideal derivation of $\mathfrak{h}$ is given by $[\mathfrak{h}, \mathfrak{h}] = \langle h_5, h_6, h_7, h_8 \rangle$. We see that $[\mathfrak{h}, \mathfrak{h}]$ is commutative derived ideal. Therefore, $\mathfrak{h}$ is 2-step solvable exact Lie algebra. Choose $\mathbb{R}^4 = [\mathfrak{h}, \mathfrak{h}] = \langle h_5, h_6, h, h_8 \rangle$. We claim as follows. Let N be spanned by 4×4 matrices N_1, N_2, N_3, N_4 where $N = \{N_1, N_2, N_3, N_4\} \subseteq M(4, \mathbb{R})$ is linearly independent and commutating 4×4 matrices such that $\mathfrak{h} = \langle h_5, h_6, h_7, h_8 \rangle \rtimes \langle N_1, N_2, N_3, N_4 \rangle = \mathbb{R}^4 \rtimes N$ where

$$N_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, N_2 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
$$N_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, N_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

To prove this, let $\mathrm{ad}|_{\mathbb{R}^4} : \langle h_1, h_2, h_3, h_4 \rangle \to \mathfrak{gl}(\mathfrak{h})$ be an adjoint representation of $\langle h_1, h_2, h_3, h_4 \rangle$ on the space $\mathfrak{h}$. Then by direct computations and by Theorem 4 we have $N_i = \mathrm{ad}(h_i)|_{\mathbb{R}^4}$ with $i = 1,2,3,4$.

1. $N_1 = \mathrm{ad}(h_1)|_{\mathbb{R}^4} = [\mathrm{ad}(h_1)h_5 \quad \mathrm{ad}(h_1)h_6 \quad \mathrm{ad}(h_1)h_7 \quad \mathrm{ad}(h_1)h_8]$

   Where :

$\mathrm{ad}(h_1)h_5 = [h_1, h_5] = (1,0,0,0)^T$

$\mathrm{ad}(h_1)h_6 = [h_1, h_6] = (0,1,0,0)^T$

$\mathrm{ad}(h_1)h_7 = [h_1, h_7] = (0,0,0,0)^T$

$\mathrm{ad}(h_1)h_8 = [h_1, h_8] = (0,0,0,0)^T.$

Namely we have $N_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Continuing computations for matrices N_2, N_3, N_4 then we have also the matrix forms N_2, N_3, N_4 as desired. In addition, we can observe that the set $N = \{N_1, N_2, N_3, N_4\}$ is linearly independent and their matrices are commutative. By direct computation of determinant of a structure matrix of $\mathfrak{h}$ then we have a Frobenius functional of $\mathfrak{h}$ is of a form $\alpha = h_5^* + h_7^*$ and its symplectic form is $\beta = \partial\alpha = h_2^* \wedge h_6^* + h_5^* \wedge h_1^* + h_7^* \wedge h_3^* + h_8^* \wedge h_4^*$.

4. Conclusion

In this research we proved that 2-step solvable Lie algebras with exact condition can be constructed by using derived ideals, linearly independent set, and commutative matrices. In this way, we give some examples of low dimensional of 2-step solvable exact Lie algebras of dimension 2,4,6, and 8.

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6. References

Alvarez, M. A., Rodríguez-Vallarte, M. C., & Salgado, G. (2018). Contact and Frobenius solvable Lie algebras with abelian nilradical. *Communications in Algebra*, *46*(10), 4344–4354. https://doi.org/10.1080/00927872.2018.1439048

Csikós, B., & Verhóczki, L. (2007). Classification of Frobenius Lie algebras of dimension ≤ 6. *Publicationes Mathematicae Debrecen*, *70*(3–4), 427–451. https://doi.org/10.5486/PMD.2007.3556

Diatta, A., & Manga, B. (2014). On Properties of Principal Elements of Frobenius Lie Algebras . *Journal of Lie Theory*, *24*(3), 849–864.

Diatta, A., Manga, B., & Mbaye, A. (2020). *On systems of commuting matrices, Frobenius Lie algebras and Gerstenhaber's Theorem*. http://arxiv.org/abs/2002.08737

Elashvili, A. G. (1983). Frobenius Lie algebras. *Functional Analysis and Its Applications*, *16*(4), 326–328. https://doi.org/10.1007/BF01077870

Gerstenhaber, M., & Giaquinto, A. (2009). The Principal Element of a Frobenius Lie Algebra. *Letters in Mathematical Physics*, *88*(1–3), 333–341. https://doi.org/10.1007/s11005-009-0321-8

Kurniadi, E., Carnia, E., & Supriatna, A. K. (2021). The construction of real Frobenius Lie algebras from non-commutative nilpotent Lie algebras of dimension. *Journal of Physics: Conference Series*, *1722*(1), 012025. https://doi.org/10.1088/1742-6596/1722/1/012025

Ooms, A. I. (1974). On Lie algebras having a primitive universal enveloping algebra. *Journal of Algebra*, *32*(3), 488–500. https://doi.org/10.1016/0021-8693(74)90154-9

Ooms, A. I. (1976). On Lie algebras with primitive envelopes, supplements. *Proceedings of the American Mathematical Society*, *58*(1), 67–67. https://doi.org/10.1090/S0002-9939-1976-0430007-6

Ooms, A. I. (1980). On frobenius lie algebras. *Communications in Algebra*, *8*(1), 13–52. https://doi.org/10.1080/00927878008822445

Ooms, A. I. (2009). Computing invariants and semi-invariants by means of Frobenius Lie algebras. *Journal of Algebra*, *321*(4), 1293–1312. https://doi.org/10.1016/j.jalgebra.2008.10.026

Pham, D. N. (2016). $\mathfrak{g}$-quasi-Frobenius Lie algebras. *Archivum Mathematicum*, *4*, 233–262. https://doi.org/10.5817/AM2016-4-233